CONSTRAINED SIMILARITY OF SURFACES BY MINIMIZING THE L^2 NORM OF THE GRADIENT OF THE SURFACE DIFFERENCE

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***Abstract**: This paper defines constrained similarity between surfaces via minimizing the L^2 norm of the gradient of the difference between the surfaces. An exact general solution is obtained for the case wherein the surfaces are given as mesh-functions defined on a uniform mesh and the imposed constraints are linear. Various examples are presented as well as a MATLAB code for the solution of one of the examples. The code could be adjusted to other cases.*

Keywords: SIMILARITY OF SURFACES, L^2 NORM GRADIENT MINIMIZATION, CONSTRAINED OPTIMIZATION

1. Introduction

Suppose a surface is given and a new surface is sought that meets a number of imposed constraints and is as similar in behaviour (shape) to the original surface as possible without necessarily being close [1] to it. Such shape optimization problems may have wide range of applications in various engineering fields [2] such as mechanics, fluid mechanics, aerodynamics, general transport phenomena, design and engineering of machines and equipment, etc. In [3] the authors have introduced constrained functional similarity between real-valued functions of one real independent variable via minimizing the H^1 semi-norm [4] of the difference between the functions. A general solution has been presented for mesh-functions under linear constraints. In [5] the definition has been extended to 2D trajectories and a general solution for the discretized case has been obtained. The similarity of functions of two and more independent variables is no less important. This work presents a definition of constrained similarity between real-valued functions of two real independent variables, i.e. similarity between surfaces. An exact general solution for mesh functions defined on a uniform mesh and subject to linear constraints is presented.

2. Constrained similarity of surfaces

Let $z^*=z^*(x,y)$ and $z=z(x,y)$ be two real-valued functions of the real independent variables $x\in[x_a, x_b]$ and $y\in[y_a, y_b]$. The functions z^* and z define two surfaces. The surface z^* will be *similar* to z, under certain given constraints, if z^* minimizes the square of the L^2 norm (Euclidean norm) of the gradient of the difference z^*-z:

$$\|\nabla z^*-\nabla z\|^2=\int_{y_a}^{y_b}\int_{x_a}^{x_b}\left(\frac{\partial z^*}{\partial x}\vec{e}_x+\frac{\partial z^*}{\partial y}\vec{e}_y-\frac{\partial z}{\partial x}\vec{e}_x-\frac{\partial z}{\partial y}\vec{e}_y\right)^2 dxdy=$$
$$=\int_{y_a}^{y_b}\int_{x_a}^{x_b}\left(\frac{\partial z^*}{\partial x}-\frac{\partial z}{\partial x}\right)^2 dxdy+\int_{y_a}^{y_b}\int_{x_a}^{x_b}\left(\frac{\partial z^*}{\partial y}-\frac{\partial z}{\partial y}\right)^2 dxdy \qquad (1)$$

and at the same time satisfies the constraints in question. In the present work only linear constraints for z^* will be considered. For example, linear combinations of functional values $z^*_{k,l}$ at certain points (x_k,y_l), integral constraints like $\int_{y_a}^{y_b}\int_{x_a}^{x_b} f(x,y)z^*(x,y)dxdy=1$, etc.

3. Exact solution for discretized surfaces under linear constraints

Suppose the intervals $x\in[x_a,x_b]$ and $y\in[y_a,y_b]$ could be partitioned by N_x and N_y mesh points into N_x-1 and N_y-1 intervals of size h, respectively. The set of points $\{(x_k,y_l)$, $x_k=x_a+(k-1)h$, $y_l=y_a+(l-1)h$, $k=1,2,\ldots,N_x$, $l=1,2,\ldots,N_y\}$ defines a uniform mesh on the rectangle. Let the function $z(x,y)$ be defined on the mesh, i.e. $\{z_{k,l}=z(x_k,y_l)$, $k=1,2,\ldots,N_x$, $l=1,2,\ldots,N_y\}$. In order to define constrained similarity between the surfaces z^* and z expression (1) is discretized using the finite differences $(z^*_{k+1,l}-z^*_{k,l})/h$, etc. for the respective derivatives $\partial z^*/\partial x$, etc. at (x_k,y_l), and the integral is replaced by a sum. The constant factors are omitted since they do not affect the minimization. Thus, the following objective function is obtained:

$$I=\sum_{k=1}^{N_x-1}\sum_{l=1}^{N_y}((z^*_{k+1,l}-z^*_{k,l})-(z_{k+1,l}-z_{k,l}))^2+$$
$$+\sum_{k=1}^{N_x}\sum_{l=1}^{N_y-1}((z^*_{k,l+1}-z^*_{k,l})-(z_{k,l+1}-z_{k,l}))^2. \qquad (2)$$

In order that the formulas, derived in [3], could be used we denote $z_{k,l}=u_{k+(l-1).Nx}$ and $z^*_{k,l}=u^*_{k+(l-1).Nx}$ for $k=1,2,..,N_x$, $l=1,2,.., N_y$ and introduce the vectors:
$u=[z_{1,1},z_{2,1},..,z_{Nx,1},z_{1,2},z_{2,2},..,z_{Nx,2},..,z_{1,Ny},z_{2,Ny},..,z_{Nx,Ny}]^T$,
$u^*=[z^*_{1,1},z^*_{2,1},..,z^*_{Nx,1},z^*_{1,2},z^*_{2,2},..,z^*_{Nx,2},..,z^*_{1,Ny},z^*_{2,Ny},..,z^*_{Nx,Ny}]^T$.
The minimum of I is sought subject to M linear constraints:

$$\sum_{i=1}^{N}A_{ji}u^*_i=c_j, \qquad j=1,2,...,M<N. \qquad (3)$$

where $N=N_xN_y$. The constraints (3) can be written in a matrix form as $Au^*=c$, where

$$A=\begin{bmatrix}A_{11} & A_{12} & . & . & . & A_{1N}\\ A_{21} & A_{22} & . & . & . & A_{2N}\\ . & . & . & . & . & .\\ A_{M1} & A_{M2} & . & . & . & A_{MN}\end{bmatrix},\quad c=\begin{bmatrix}c_1\\ c_2\\ .\\ c_M\end{bmatrix} \qquad (4)$$

and u^* is the $N\times1$ column-vector of the unknowns. To find the minimum of I subject to constraints (3) the Lagrange's method of the undetermined coefficients [6] is used. First, the Lagrangian

$$J=I+\sum_{j=1}^{M}\left(\lambda_j(c_j-\sum_{i=1}^{N}A_{ji}u^*_i)\right) \qquad (5)$$

is introduced, where λ_j, $j=1,2,\ldots,M$, are the Lagrange's undetermined coefficients. Then, the derivatives of J with respect to u^*_i, $i=1,2,\ldots,N$ are equated to zero, and the obtained system of equations is written in a matrix form as:

$$\bar{L}u^*=\bar{L}u-\frac{1}{2}A^T\lambda, \qquad (6)$$

where λ is the $M\times1$ column-vector of the undetermined coefficients and $\bar{L}$ is the $N\times N$ matrix ($N=N_xN_y$) given below:

$$\bar{L}=\left|\begin{array}{ccccc}
\overbrace{\begin{matrix}-2&1&0&0\\1&-3&1&0\\0&1&-3&1\\&\cdot&\cdot&\cdot\\&1&-3&1\\&0&1&-2\end{matrix}}^{N_x} & \overbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\\&&\cdot\\&1&0\\&0&1\end{matrix}}^{N_x} & \overbrace{\quad\quad\quad}^{N_x} & \overbrace{\quad\quad\quad}^{N_x} & \overbrace{\quad\quad\quad}^{N_x}\\
\begin{matrix}1&0&0\\0&1&0\\0&0&1\\&&\cdot\\&1&0\\&0&1\end{matrix} & \begin{matrix}-3&1&0&0\\1&-4&1&0\\0&1&-4&1\\&\cdot&\cdot&\cdot\\&1&-4&1\\&0&1&-3\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\&&\cdot\\&1&0\\&0&1\end{matrix} & & \\
 & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\&&\cdot\\&1&0\\&0&1\end{matrix} & \begin{matrix}-3&1&0&0\\1&-4&1&0\\0&1&-4&1\\&\cdot&\cdot&\cdot\\&1&-4&1\\&0&1&-3\end{matrix} & & \\
 & & & \begin{matrix}-3&1&0&0\\1&-4&1&0\\0&1&-4&1\\&\cdot&\cdot&\cdot\\&1&-4&1\\&0&1&-3\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\&&\cdot\\&1&0\\&0&1\end{matrix}\\
 & & & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\&&\cdot\\&1&0\\&0&1\end{matrix} & \begin{matrix}-2&1&0&0\\1&-3&1&0\\0&1&-3&1\\&\cdot&\cdot&\cdot\\&1&-3&1\\&0&1&-2\end{matrix}
\end{array}\right| \qquad (7)$$

In order to remove the singularity of $\bar{L}$ we need to add the equations for the constraints to equations (6). For this reason the matrix A is augmented with $N–M$ rows of zeros to get the $N\times N$ matrix $\bar{A}$. Correspondingly, the column-vector c is augmented with $N–M$ zeros to get the $N\times 1$ column-vector $\bar{c}$. Now, the results for u^* and λ derived in [3] can be used:

$$u^* = u - (\bar{L}+\bar{A})^{-1}\left(\frac{1}{2}A^T\lambda + \bar{A}u - \bar{c}\right), \qquad (8)$$

$$\lambda = 2\left(A(\bar{L}+\bar{A})^{-1}A^T\right)^{-1}\left(Au - c - A(\bar{L}+\bar{A})^{-1}(\bar{A}u-\bar{c})\right), \quad (9)$$

where $\bar{L}$ is defined in (7). The right-hand side of (9) contains only known quantities. Once the column-vector λ is calculated it is substituted into (8) and u^*_i, $i=1,2,\dots,N$ are obtained. Then u^*_i are converted to $z^*_{k,l}$ and the sought surface is found.

4. Results

In this paragraph two examples are presented with two types of constraints: boundary and integral constraints.

Example 1

Consider the surface $z=(x,y)$ defined by $\{z_{k,l}=-x_k^2+y_l^2$, $x_k=x_a+h(k-1)$, $y_l=y_a+h(l-1)$, $k=1,2,\dots, N_x$, $l=1,2,\dots, N_y\}$ on a rectangular uniform mesh with $x_a=-2$, $x_b=2$, $y_a=-2$, $y_b=2$. The step-size of the mesh is $h=0.2$, hence $N_x=21$ and $N_y=21$. Using (8) and (9), the surface z^*, i.e. $\{z^*_{k,l}$, $k=1,2,\dots, N_x$, $l=1,2,\dots, N_y\}$, similar to z and satisfying the following constraints at the boundaries (along the perimeter of the rectangle)

$$z^*_{k,l} = z_{k,l} + \Delta z_{k,l} \qquad (10)$$

where $k=1,2,\dots, N_x$ for $l=1$ and $l=N_y$, and $l=2,\dots, N_y-1$ for $k=1$ and $k=N_x$ (all the mesh-points at the boundary), is found for several $\Delta z_{k,l}$ (see Fig.1).

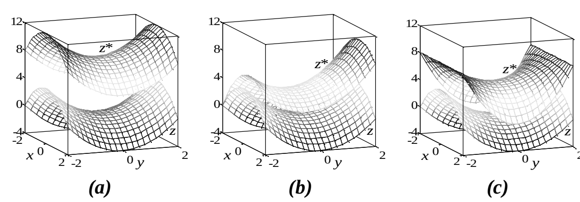

Fig.1. *The original surface z and the similar to it surface z* satisfying constraints (10) for* ***(a)*** $\Delta z_{k,l}=8$; ***(b)*** $\Delta z_{k,l}=x_k+y_l+4$; *and* ***(c)*** $\Delta z_{k,l}=x_k^2+y_l^2$.

Example 2

Consider the surface $z=(x,y)$ defined by $\{z_{k,l}=0$, $x_k=x_a+h(k-1)$, $y_l=y_a+h(l-1)$, $k=1,2,\dots, N_x$, $l=1,2,\dots, N_y\}$ on a rectangular uniform mesh with $x_a=-2$, $x_b=2$, $y_a=-2$, $y_b=2$. The step-size of the mesh is $h=0.2$, hence $N_x=21$ and $N_y=21$. Using (8) and (9), the surface z^*, i.e. $\{z^*_{k,l}$, $k=1,2,\dots, N_x$, $l=1,2,\dots, N_y\}$, similar to z and satisfying the following constraints at the boundary (along the perimeter of the rectangle)

$$z^*_{k,l} = z_{k,l} + x_k^2 + y_l^2 \qquad (11)$$

where $k=1,2,\dots, N_x$ for $l=1$ and $l=N_y$, and $l=2,\dots, N_y-1$ for $k=1$ and $k=N_x$ (all the mesh-points at the boundary), and the integral constraint

$$\sum_{k=1}^{N_x}\sum_{l=1}^{N_y} z^*_{k,l} = \sum_{k=1}^{N_x}\sum_{l=1}^{N_y} z_{k,l} + \Delta V \qquad (12)$$

is found for several values of ΔV (see Fig.2).

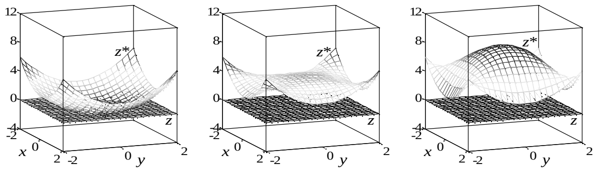

Fig.2. *The original surface z and the similar to it surface z* satisfying constraints (11) and (12) for* ***(a)*** *ΔV=600;* ***(b)*** *ΔV=1600; and* ***(c)*** *ΔV=2400.*

5. Conclusion

This work defined constrained similarity between surfaces via minimizing the L^2 norm of the gradient of the difference between the surfaces. An exact general solution was obtained for discreized surfaces under linear constraints. The results agree with what is expected from similarity of surfaces under constraints.

6. Appendix

In this appendix a MATLAB code for solving Example 1(c) is presented. The variables `A_`, `c_`, and `L_` are used for $\overline{A}$, $\overline{c}$, and $\overline{L}$, while `zs`, `us`, `dz`, and `du` are used for z^*, u^*, Δz, and Δu, respectively. The variable `lambda` is used for λ. To define the needed vectors and matrices first the corresponding vectors and matrices composed of zeros and having the required size are defined. The obtained graph needs to be rotated to be seen from aside.

```
function main

 Nx=21; Ny=21; xa=-2; ya=-2; h=0.2;
 M=2*Nx+2*Ny-4; N=Nx*Ny;

 x=zeros(Nx,1); y=zeros(Ny,1);
 for k=1:Nx
  x(k)=xa+h*(k-1);
 end
 for l=1:Ny
  y(l)=ya+h*(l-1);
 end

 z=zeros(Nx,Ny); u=zeros(N,1);
 dz=zeros(Nx,Ny); du=zeros(N,1); i=1;
 for l=1:Ny
  for k=1:Nx
   z(k,l)=-x(k)*x(k)+y(l)*y(l); u(i)=z(k,l);
   dz(k,l)=x(k)*x(k)+y(l)*y(l); du(i)=dz(k,l);
   i=i+1;
  end
 end

 A=zeros(M,N); c=zeros(M,1); j=1;
 for k=1:Nx
  i=k; A(j,i)=1; c(j)=u(i)+du(i); j=j+1;
 end
 for l=2:Ny-1
  i=Nx*(l-1)+1; A(j,i)=1; c(j)=u(i)+du(i);
  j=j+1;
  i=Nx*(l-1)+Nx; A(j,i)=1; c(j)=u(i)+du(i);
  j=j+1;
 end
 for k=1:Nx
  i=Nx*(Ny-1)+k; A(j,i)=1; c(j)=u(i)+du(i);
  j=j+1;
 end

 A_=zeros(N,N); c_=zeros(N,1);
 for j=1:M
  c_(j)=c(j);
  for i=1:N
   A_(j,i)=A(j,i);
  end
 end

 L_=zeros(N,N);
 L_(1,1)=-2; L_(Nx,Nx)=-2;
 L_(N-Nx+1,N-Nx+1)=-2; L_(N,N)=-2;
 for n=2:(Nx-1)
  L_(n,n)=-3; L_(N-Nx+n,N-Nx+n)=-3;
  L_(n+1,n)=1; L_(N-Nx+n+1,N-Nx+n)=1;
  L_(n-1,n)=1; L_(N-Nx+n-1,N-Nx+n)=1;
  L_(n+Nx,n)=1; L_(N-Nx+n-Nx,N-Nx+n)=1;
 end
 L_(2,1)=1; L_(Nx-1,Nx)=1;
 L_(1+Nx,1)=1; L_(Nx+Nx,Nx)=1;
 L_(N-Nx+2,N-Nx+1)=1; L_(N-Nx+1-Nx,N-Nx+1)=1;
 L_(N-1,N)=1; L_(N-Nx,N)=1;
 for n=Nx+1:N-Nx
  L_(n,n)=-4; L_(n+1,n)=1; L_(n-1,n)=1;
  L_(n+Nx,n)=1; L_(n-Nx,n)=1;
 end
 for n=1:Ny-2
  L_(Nx*n+1,Nx*n+1)=-3;
  L_(Nx*n+Nx,Nx*n+Nx)=-3;
  L_(Nx*n,Nx*n+1)=0; L_(Nx*n+Nx+1,Nx*n+Nx)=0;
 end

 H=inv(L_+A_); d=A_*u-c_;

 lambda=(A*H*A')\(A*u-c-A*H*d)*2;
 us=u-H*(A'*lambda/2+d);

 i=1;

 for l=1:Ny
  for k=1:Nx
   zs(k,l)=us(i);
   i=i+1;
  end
 end

 hold on; surface(x,y,z'); surface(x,y,zs');

end
```

7. References

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